Today: Exponential Growth (Differential Equations)

$f(n) =$ # of grains of rice on nth square of board.

$f(1)=1,\ f(2)=2,\ f(3)=4,\ f(4)=8, \ldots$

In general $f(n+1) = 2 \cdot f(n) \rightsquigarrow \underbrace{f(n) = 2^{n-1}}_{\text{exponential growth}}$

Exponential

$f(n) = 2^n$

$f(n+1) - f(n) =$

$2^{n+1} - 2^n = 2^n = f(n)$

"At the nth stage, grow by $f(n)$"

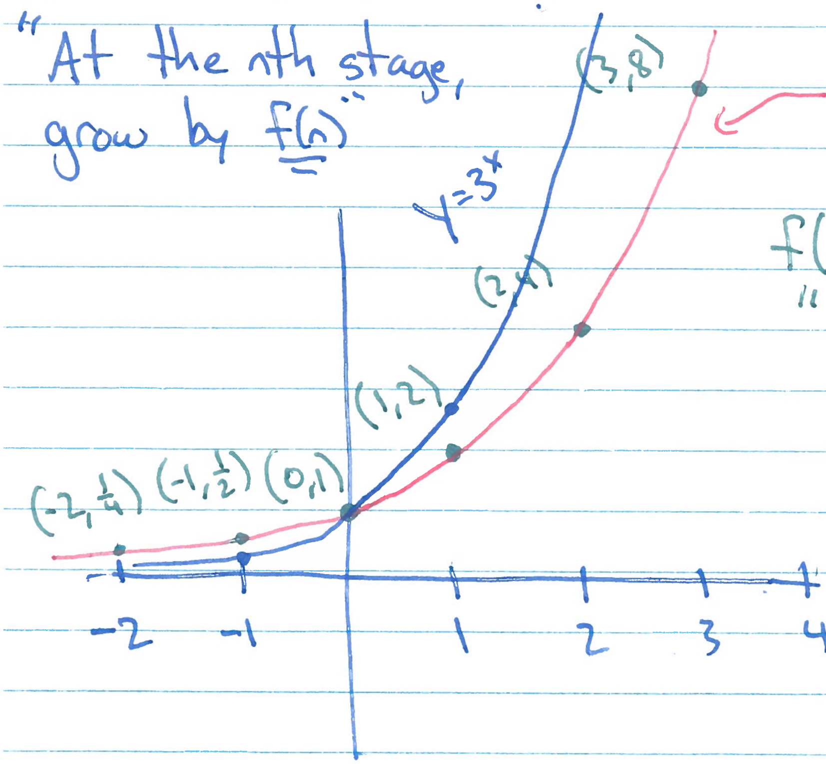
Quadratic

$f(n) = n^2$

$f(n+1) - f(n) = (n+1)^2 - n^2 = 2n+1$

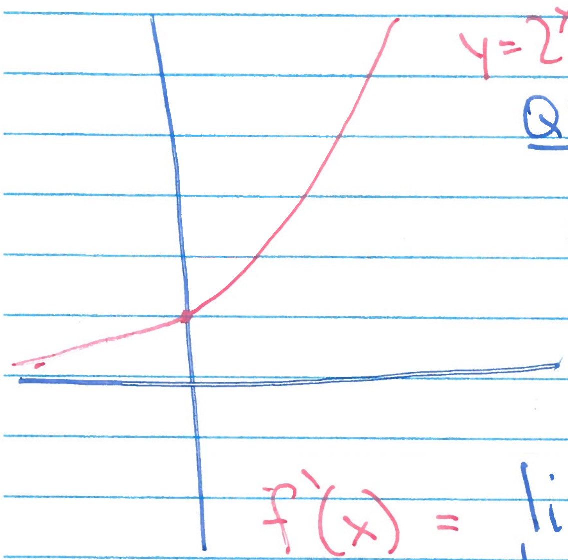
"At the nth stage, grow by $2n+1$"

$y = 2^x$. $2^{x_1+x_2} = 2^{x_1} \cdot 2^{x_2}$



$f(n) = 2^n$

"Each value is 2(previous value)"



Q: What is the derivative?

Remember $\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$

$$f'(x) = \lim_{h\to 0} \frac{2^{x+h} - 2^x}{h} = \lim_{h \to 0} \frac{2^x \cdot 2^h - 2^x}{h}$$

$$= \lim_{h\to 0} \frac{2^x(2^h - 1)}{h}$$

$$= \underbrace{2^x}_{f(x)} \cdot \underbrace{\lim_{h \to 0} \frac{2^h - 1}{h}}_{\text{Constant}}$$

The derivative of

$f(x) = a^x$ (a a constant)

is $a^x \cdot C_a$

where $C_a = \lim_{h\to 0} \frac{a^h - 1}{h}$

Experimentally: $C_a = 1$ when $a \approx 2.718\ldots =: e$

Def: e is the number s.t. $f(x) = e^x \rightsquigarrow f'(x) = e^x$

Ce^x is a solution to the DE $\boxed{f(x) = f'(x)}$

$$\frac{d}{dx}(a^x) = \ln(a) \cdot a^x$$

Another perspective.

e^x & $\ln(x)$ are inverse functions

meaning $e^{\ln(x)} = x$, and $\ln(e^x) = x$

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{\ln(a)}\right)^x = \frac{d}{dx}\left(e^{\ln(a)\cdot x}\right)$$

$$= e^{\ln(a)\cdot x} \cdot \ln(a)$$

$$= a^x \cdot \ln(a)$$

Examples of exponential growth:

- Colony of bacteria
- Populations
- Decay of a substance in the body.

(Exponential Decay)

Algebraic Equation	Differential Equations
$x^2 = 7$	$f(x) = f'(x)$
} square root	}
$x = \pm\sqrt{7}$] Solution is a number	$f(x) = e^x$
	Solution is a function
$3x^2 + 2x - 7 = 0$	
} Quadratic formula	$3f'(x) + f'(x)^2 = 2f(x) + 3$
	$f''(x) - f'(x) + 2f(x) = 0$
	A relationship between the function & its derivatives.

$$\frac{d}{dx}(e^x) = e^x$$

Natural log

$\ln(2) \approx 0.691$
$\ln(e) = 1$

$$\frac{d}{dx}(a^x) = C_a a^x = \ln(a)\, a^x$$